The Gysin sequence for $\mathbb{S}^3$-actions on manifolds*

José Ignacio Royo Prieto† Martintxo Saralegi-Aranguren‡

Universidad del País Vasco/
Euskal Herriko Unibertsitatea

Université d'Artois

Abstract

Given a smooth action of $\mathbb{S}^3$ on a manifold M, we are interested in the relationship between the cohomologies of M and $M/\mathbb{S}^3$. If the action is free, we have indeed a principal $\mathbb{S}^3$-bundle, and this relationship is described by the classical Gysin sequence, which also exists when the action is semi-free (i.e., fixed points are allowed) [3]. In this work, we obtain a Gysin sequence for the case of a general smooth action. An exotic term appears, and we show that it is an obstruction for the duality of the second term of the de Rham spectral sequence associated to the action.

Let us consider a smooth action $\Phi\colon \mathbb{S}^3 \times M \to M$. The cohomology of the manifold M can be computed by means of the de Rham spectral sequence. The main point is the understanding of the second term $E_2^{p,q}$ of this spectral sequence. When the action is (almost) free this second term is just[1] $H^p\left(M/\mathbb{S}^3\right) \otimes H^q\left(\mathbb{S}^3\right)$. When the action Φ is semi-free, we have the Gysin sequence:

$$\cdots \longrightarrow H^i(M) \xrightarrow{②} H^{i-3}\left(M/\mathbb{S}^3, M^{\mathbb{S}^3}\right) \xrightarrow{③} H^{i+1}\left(M/\mathbb{S}^3\right) \xrightarrow{①} H^{i+1}(M) \longrightarrow \cdots,$$

where the morphism ① is induced by the natural projection $\pi\colon M \to M/\mathbb{S}^3$, the morphism ② is induced by the integration along the fibers of π and the morphism ③ is the multiplication by the *Euler class* $[e] \in \boldsymbol{H}^4_{\bar{4}}\left(M/\mathbb{S}^3\right)$ (cf. [6]).

The main goal of this work is to extend this result to any smooth action of $\mathbb{S}^3$. We obtain the following Gysin sequence (cf. Theorem 2.3 and paragraph 2.5)

$$\cdots \longrightarrow H^i(M) \xrightarrow{②} H^{i-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{i-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \xrightarrow{③} H^{i+1}\left(M/\mathbb{S}^3\right) \xrightarrow{①} H^{i+1}(M) \longrightarrow \cdots$$

where

2010 Mathematical Subject Classification. Primary 57R19; Secondary 57R30, 57S15.

Key Words and Phrases: Gysin sequence, basic cohomology, $\mathbb{S}^3$-actions, controlled forms.

*This work has been partially supported by the projects MTM2007-66262 (Spanish Department of Science and Technology) and EHU09-04 (University of the Basque Country).

†Departamento de Matemática Aplicada/Matematika Aplikatua Saila, UPV/EHU. Alameda de Urquijo s/n. 48013 Bilbao, SPAIN. *joseignacio.royo@ehu.es*.

‡Université d'Artois, Laboratoire de Mathématiques de Lens EA 2462. Fédération CNRS Nord-Pas-de-Calais FR 2956. Faculté des Sciences Jean Perrin. Rue Jean Souvraz, S.P. 18. F-62 300 Lens, France. *saralegi@euler.univ-artois.fr*.

[1] In this work, $H^*(X)$ stands for the singular cohomology of the space X with real coefficients.

- Σ is the subset of points of M whose isotropy group is infinite;
- the $\mathbb{Z}_2$-action is induced by $j \in \mathbb{S}^3$ and
- $(-)^{-\mathbb{Z}_2}$ denotes the subspace of antisymmetric elements (cf. (6)).

A word about the duality of the second term $E_2^{s,t}$ of the de Rham spectral sequence associated to the action. When Φ is free, we have the duality $E_2^{s,t} \cong E_2^{n-s,3-q}$, where $n = \dim M/\mathbb{S}^3$. This property is lost when singularities appear, but duality is recovered in the semi-free case by using intersection cohomology. In fact, we have ${}_{\overline{p}}E_2^{s,t} \cong {}_{\overline{q}}E_2^{n-s,3-t}$ where $\overline{p}$ and $\overline{q}$ are dual perversities [6] (see [5] for the circle case).

For a general Φ, the exotic term $\left(H^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$ we have found may be interpreted as an obstruction for extending this duality as follows. The calculations made in this work show that ${}_{\overline{0}}E_2^{s,0} = H^s\left(M/\mathbb{S}^3\right)$, ${}_{\overline{0}}E_2^{s,2} = \left(H^s\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$, ${}_{\overline{0}}E_2^{s,3} = H^s\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right)$ and ${}_{\overline{p}}E_2^{s,1} = 0$ for any perversity $\overline{p}$, which makes duality impossible when the exotic term is not 0, as happens in 2.4. This example also shows that this duality, which holds for regular Riemannian foliations [1], does not work for singular Riemannian foliations, even considering perversities.

In the sequel M is a connected, second countable, Haussdorff, without boundary and smooth (of class C^∞) manifold. We fix a smooth action $\Phi \colon \mathbb{S}^3 \times M \to M$.

1. Stratifications and differential forms.

We describe the stratification arising from the action. We also introduce the controlled differential forms, defined by Verona, in order to compute the singular cohomology in this context.

1.1. Thom-Mather structure.

There are three possibilities for the dimension of the isotropy subgroup[2] $\mathbb{S}^3_x$ of a point $x \in M$, namely: 0,1 and 3. So, we have the dimension-type filtration

$$F = \left\{x \in M \;\middle|\; \dim \mathbb{S}^3_x = 3\right\} \subset \Sigma = \left\{x \in M \;\middle|\; \dim \mathbb{S}^3_x \geq 1\right\} \subset M = \left\{x \in M \;\middle|\; \dim \mathbb{S}^3_x \geq 0\right\}.$$

In this section, we describe the geometry of the triple (M, Σ, F). The subset Σ is not necessarily a manifold, but subsets $F = M^{\mathbb{S}^3}$, $\Sigma \backslash F = \left\{x \in M \;\middle|\; \dim \mathbb{S}^3_x = 1\right\}$ and $M\backslash\Sigma = \left\{x \in M \;\middle|\; \dim \mathbb{S}^3_x = 0\right\}$ are proper invariant submanifolds[3] of M. So, we can consider $\tau_0 \colon T_0 \to F$ and $\tau_1 \colon T_1 \to \Sigma\backslash F$ two invariant tubular neighborhoods in M. Over each connected component, the structure group is the orthogonal group. Associated to these tubular neighborhoods we have the following maps ($k = 0, 1$):

⇝ The *radius map* $\nu_k \colon T_k \to [0, \infty[$, defined fiberwiselly by $u \mapsto ||u||$. It is an invariant smooth map.

⇝ The *dilatation map* $\partial_k \colon [0, \infty[\times T_k \to T_k$, defined fiberwiselly by $(t, u) \mapsto t \cdot u$. It is a smooth equivariant map.

The family of tubular neighborhoods $\mathfrak{T}_M = \{T_0, T_1\}$ is a *Thom-Mather system* when:

$$\text{(TM)} \left\{ \begin{array}{l} \tau_0 = \tau_0 \circ \tau_1 \\ \nu_0 = \nu_0 \circ \tau_1 \end{array} \right\} \text{ on } T_0 \cap T_1 = \tau_1^{-1}(T_0 \cap (\Sigma\backslash F)).$$

Lemma 1.2 *Thom-Mather systems exist.*

Proof. We fix an invariant tubular neighborhood $\tau_0 \colon T_0 \to F$. It exists since F is an invariant closed submanifold of M. Since the isotropy subgroup of any point of F is the whole $\mathbb{S}^3$, we can find[4] an atlas

[2] We refer the reader to [3] for the notions related with compact Lie group actions, such as isotropy, invariant tubular neighborhoods,...

[3] In fact, these manifolds may have connected components with different dimensions.

[4] For each connected component of F.

$\mathcal{A} = \left\{\varphi \colon U \times \mathbb{R}^n \to \tau_0^{-1}(U)\right\}$ of τ_0, having $O(n)$ as structure group, and an orthogonal action $\Psi \colon \mathbb{S}^3 \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi(x, \Psi(g, v)) = \Phi(g, \varphi(x, v)) \quad \forall x \in U, \forall v \in \mathbb{R}^n \text{ and } \forall g \in \mathbb{S}^3. \tag{1}$$

We write $\tau_0' \colon S_0 \to F$ the restriction of τ_0, where S_0 is the submanifold $\nu_0^{-1}(1)$. It is a fiber bundle. The restriction $\tau_0'' \colon (S_0 \cap (\Sigma \backslash F)) \to F$ is also a fiber bundle whose induced atlas is $\mathcal{A}'' = \left\{\varphi \colon U \times \mathbb{S}^{n-1}_\Sigma \to \tau_0''^{-1}(U)\right\}$, where $\mathbb{S}^{n-1}_\Sigma = \left\{w \in \mathbb{S}^{n-1} \mid \dim \mathbb{S}^3_w = 1\right\}$.

The map $\mathfrak{L}_0 \colon T_0 \backslash F \to S_0 \times]0, \infty[$, defined by $\mathfrak{L}_0(x) = \left(\partial_0\left(\nu_0(x)^{-1}, x\right), \nu_0(x)\right)$, is an equivariant diffeomorphism. Under $\mathfrak{L}_0$:

⇝ the map τ_0 becomes $(y, t) \mapsto \tau_0'(y)$,

⇝ the map ν_0 becomes $(y, t) \mapsto t$, and

⇝ the manifold $T_0 \cap (\Sigma \backslash F)$ becomes $(S_0 \cap (\Sigma \backslash F)) \times]0, \infty[$.

Since the structure group of τ_0' is a compact Lie group, condition (1) allows us to construct an invariant Riemannian metric μ_0 on S_0 such that the fibers of τ_0' are totally geodesic submanifolds and $(T(S_0 \cap (\Sigma \backslash F)))^\perp \subset \ker\left(\tau_0'\right)_*$. Then, if we consider the associated tubular neighborhood $\tau_1' \colon T_1' \to S_0 \cap (\Sigma \backslash F)$ we have $\tau_0' \circ \tau_1' = \tau_0'$.

We can construct now an invariant Riemannian metric μ on $M \backslash F$ such that under $\mathfrak{L}_0$:

⇝ the metric μ becomes $\mu_0 + dr^2$ on $S_0 \times]0, \infty[$.

We consider the associated tubular neighborhood $\tau_1 \colon T_1 \to \Sigma \backslash F$. Verification of the property (TM) must be done on $T_0 \cap T_1$, where using $\mathfrak{L}_0$, we get:

⇝ $T_0 \cap T_1$ becomes $T_1' \times]0, \infty[$.

⇝ τ_1 becomes $(y, t) \mapsto (\tau_1'(y), t)$.

A straightforward calculation gives (TM) and ends the proof. ♣

We fix a such system $\mathfrak{T}_M$. For each $k \in \{0, 1\}$, we shall write $D_k \subset M$ the open subset $\nu_k^{-1}([0, 1[)$ and call it the *soul* of the tubular neighborhood τ_k. We shall write $\Delta_0 = D_0 \cap \Sigma$.

1.3. Verona's differential forms. As it is shown in [7], the singular cohomology of M can be computed by using differential forms on $M \backslash \Sigma$. This is the tool we use in this work. The complex of *controlled forms* (or *Verona's forms*) of M is defined by

$$\Omega^*_v(M) = \left\{ \omega \in \Omega^*(M \backslash \Sigma) \;\middle|\; \exists \omega_1 \in \Omega^*(\Sigma \backslash F) \text{ and } \omega_0 \in \Omega^*(F) \text{ with } \left\{ \begin{array}{l} (a)\ \tau_1^* \omega_1 = \omega \ \text{ on } D_1 \backslash \Sigma \\ (b)\ \tau_0^* \omega_0 = \omega \ \text{ on } D_0 \backslash \Sigma \\ (c)\ \tau_0^* \omega_0 = \omega_1 \ \text{ on } \Delta_0 \backslash F \end{array} \right\} \right\}.$$

Following [7] we know that the cohomology of the complex $\Omega^*_v(M)$ is the singular cohomology $H^*(M)$.

We also use in this work the complex $\Omega^*_v(\Sigma) = \left\{\gamma \in \Omega^*(\Sigma \backslash F) \mid \exists \gamma_0 \in \Omega^*(F) \text{ with } \tau_0^* \gamma_0 = \gamma \ \text{ on } \Delta_0 \backslash F\right\}$ and the *relative complexes* $\Omega^*_v(M, \Sigma) = \{\omega \in \Omega^*_v(M) \mid \omega_1 \equiv 0\}$ and $\Omega^*_v(\Sigma, F) = \{\gamma \in \Omega^*_v(\Sigma) \mid \gamma_0 \equiv 0\}$.

Since M is a manifold, controlled forms are in fact differential forms on M.

Lemma 1.4 *Any controlled form of M is the restriction of a differential form of M.*

Proof. First, we construct a section σ of the restriction $\rho \colon \Omega^*_v(M) \to \Omega^*_v(\Sigma)$ defined by $\rho(\omega) = \omega_1$. Let us consider a smooth function $f \colon]0, \infty[\to [0, 1]$ verifying $f \equiv 0$ on $[3, \infty[$ and $f \equiv 1$ on $]0, 2]$. Notice that the compositions $f \circ \nu_0 \colon M \to [0, 1]$ and $f \circ \nu_1 \colon M \backslash F \to [0, 1]$ are smooth invariant maps. So, for each $\gamma \in \Omega^*_v(\Sigma)$ we have

$$\sigma(\gamma) = (f \circ \nu_0) \cdot \tau_0^* \gamma_0 + (1 - (f \circ \nu_0)) \cdot (f \circ \nu_1) \tau_1^* \gamma \in \Omega^*(M). \tag{2}$$

This differential form is a controlled form since

(a) Since $(f\circ\nu_1)\equiv 1$ on D_1, $(f\circ\nu_0)\equiv 0$ on $M\backslash T_0$ and (TM) then we have

$$\sigma(\gamma)=(f\circ\nu_0)\cdot\tau_1^*\tau_0^*\gamma_0+(1-(f\circ\nu_0))\cdot\tau_1^*\gamma=\tau_1^*\left((f\circ\nu_0)\cdot\tau_0^*\gamma_0+(1-(f\circ\nu_0))\cdot\gamma\right)$$

on $D_1\backslash\Sigma$. This gives $(\sigma(\gamma))_1=(f\circ\nu_0)\cdot\tau_0^*\gamma_0+(1-(f\circ\nu_0))\cdot\gamma$. Since $\tau_0^*\gamma_0=\gamma$ on $\Delta_0\backslash F$ then $(\sigma(\gamma))_1=(f\circ\nu_0)\cdot\gamma+(1-(f\circ\nu_0))\cdot\gamma=\gamma$.

(b) Since $(f\circ\nu_0)\equiv 1$ on D_0 then we have $\sigma(\gamma)=\tau_0^*\gamma_0$ on $D_0\backslash\Sigma$. This gives $(\sigma(\gamma))_0=\gamma_0$.

(c) We have $(\sigma(\gamma))_1=\gamma=\tau_0^*\gamma_0=\tau_0^*(\sigma(\gamma))_0$ on $\Delta_0\backslash F$.

This map σ is a section of ρ since $\rho(\sigma(\gamma))=(\sigma(\gamma))_1=\gamma$.

In particular, $\rho(\omega-\sigma(\rho(\omega)))=0$ for each $\omega\in\Omega^*_{_V}(M)$. As $\sigma(\rho(\omega))\in\Omega^*(M)$ (cf. (2)) and coincides with ω in the open set $(D_0\cup D_1)\backslash\Sigma$ we conclude that ω can be extended to M. ♣

1.5. Invariant forms. Denote by $\mathfrak{X}_\Phi(M)$ the subbundle of TM formed by the vector fields of M tangent to the orbits of Φ. A controlled form ω of M is an *invariant form* when $L_X\omega=0$ for each $X\in\mathfrak{X}_\Phi(M)$. The complex of invariant forms is denoted by $\underline{\Omega}^*_{_V}(M)$. The inclusion $\underline{\Omega}^*_{_V}(M)\hookrightarrow\Omega^*_{_V}(M)$ induces an isomorphism in cohomology. This a standard argument based on the fact that $\mathbb{S}^3$ is a connected compact Lie group (cf. [4, Theorem I, Ch. IV, vol. II]). So,

$$H^*\left(\underline{\Omega}^\cdot_{_V}(M)\right)=H^*\left(\Omega^\cdot_{_V}(M)\right)=H^*(M). \tag{3}$$

1.6. Basic forms. A controlled form ω of M is a *basic form* when $i_X\omega=i_Xd\omega=0$ for each $X\in\mathfrak{X}_\Phi(M)$. The complex of the basic forms is denoted by $\Omega^*_{_V}\left(M/\mathbb{S}^3\right)$. In a similar fashion we define $\Omega^*_{_V}\left(\Sigma/\mathbb{S}^3\right)$. In this work, we shall use the following relative versions of these complexes: $\Omega^*_{_V}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)=\Omega^*_{_V}\left(M/\mathbb{S}^3\right)\cap\Omega^*_{_V}(M,\Sigma)$, as well as $\Omega^*_{_V}\left(\Sigma/\mathbb{S}^3,F\right)=\Omega^*_{_V}\left(\Sigma/\mathbb{S}^3\right)\cap\Omega^*_{_V}(\Sigma,F)$.

Lemma 1.7

$$H^*\left(\Omega^\cdot_{_V}\left(M/\mathbb{S}^3\right)\right)=H^*\left(M/\mathbb{S}^3\right)\qquad\textit{and}\qquad H^*\left(\Omega^\cdot_{_V}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)\right)=H^*\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right).$$

Proof. The orbit space $M/\mathbb{S}^3$ is a stratified pseudomanifold. The family of tubular neighborhoods $\mathfrak{T}_{M/\mathbb{S}^3}=\{\pi(T_0),\pi(T_1)\}$ is a *Thom-Mather system*. Here, $\pi\colon M\to M/\mathbb{S}^3$ denotes the canonical projection. Using this projection, we identify the complex of controlled forms of $M/\mathbb{S}^3$ with $\Omega^\cdot_{_V}\left(M/\mathbb{S}^3\right)$, and the same holds for Σ.

Since $H^*\left(\Omega^\cdot_{_V}(X)\right)=H^*(X)$ for any stratified pseudomanifold X, then $H^*\left(\Omega^\cdot_{_V}\left(M/\mathbb{S}^3\right)\right)=H^*\left(M/\mathbb{S}^3\right)$ and $H^*\left(\Omega^\cdot_{_V}\left(\Sigma/\mathbb{S}^3\right)\right)=H^*\left(\Sigma/\mathbb{S}^3\right)$ (cf. [7]). In fact, the orbit spaces $M/\mathbb{S}^3$ and $\Sigma/\mathbb{S}^3$ are triangulable [8], and by [9], both of them possess good coverings. Moreover, any open covering of $M/\mathbb{S}^3$ (resp. $\Sigma/\mathbb{S}^3$) possesses a subordinated partition of unity made up of controlled functions. So, we can proceed as in [2] and construct a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^p\left(\Omega^\cdot_{_V}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)\right) \arrow[r] & H^p\left(\Omega^\cdot_{_V}\left(M/\mathbb{S}^3\right)\right) \arrow[r] \arrow[d, "f_M"] & H^p\left(\Omega^\cdot_{_V}\left(\Sigma/\mathbb{S}^3\right)\right) \arrow[r] \arrow[d, "f_\Sigma"] & H^{p+1}\left(\Omega^\cdot_{_V}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)\right) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^p\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right) \arrow[r] & H^p\left(M/\mathbb{S}^3\right) \arrow[r] & H^p\left(\Sigma/\mathbb{S}^3\right) \arrow[r] & H^{p+1}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical arrows are isomorphisms and the horizontal rows are the long exact sequences associated to the pair $(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3)$. This gives $H^*\left(\Omega^{\cdot}_{v}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right)\right) = H^*\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right)$[5]. ♣

2. Gysin sequence.

We construct the long exact sequence associated to the action $\Phi\colon \mathbb{S}^3 \times M \to M$ relating the cohomology of M and $M/\mathbb{S}^3$. First of all, we shall use strongly that Φ is *almost free*[6] in $M\backslash\Sigma$ to get a better description of the controlled forms of M.

2.1. Decomposition of a differential form. We fix $\{u_1, u_2, u_3\}$ a basis of the Lie algebra of $\mathbb{S}^3$ with $[u_1, u_2] = u_3$, $[u_2, u_3] = u_1$ and $[u_3, u_1] = u_2$. We denote by $X_i \in \mathfrak{X}_{\Phi}(M)$ the *fundamental vector field* associated to u_i, $i = 1, 2, 3$.

We endow $M\backslash\Sigma$ with an $\mathbb{S}^3$-invariant Riemannian metric μ_0, which exists because $\mathbb{S}^3$ is compact. We also fix a bi-invariant Riemannian metric ν on the Lie group $\mathbb{S}^3$. Consider now the μ_0-orthogonal $\mathbb{S}^3$-invariant decomposition $T(M\backslash\Sigma) = \mathcal{D} \oplus \xi$, where $\mathcal{D}$ is the distribution generated by Φ. Since the action Φ is almost free on $M\backslash\Sigma$, for each point $x \in M\backslash\Sigma$, the family $\{X_1(x), X_2(x), X_3(x)\}$ is a basis of $\mathcal{D}_x$. We define the $\mathbb{S}^3$-Riemannian metric μ on $M\backslash\Sigma$ by putting

$$\mu(w_1, w_2) = \begin{cases} \mu_0(w_1, w_2) & \text{if } w_1, w_2 \in \xi_x \\ 0 & \text{if } w_1 \in \xi_x, w_2 \in \mathcal{D}_x \\ \delta_{i,j} & \text{if } w_1 = X_i(x), w_2 = X_j(x) \end{cases}$$

We denote by $\chi_i = i_{X_i}\mu \in \Omega^1(M\backslash\Sigma)$ the *characteristic form* associated to u_i, $i = 1, 2, 3$. Since $\chi_j(X_i) = \mu(X_i, X_j) = \delta_{ij}$, each differential form $\omega \in \Omega^*(M\backslash\Sigma)$ possesses a unique writing,

$$\omega = {}_0\omega + \sum_{p=1}^{3} \chi_p \wedge {}_p\omega + \sum_{1\le p<q\le 3} \chi_p \wedge \chi_q \wedge {}_{pq}\omega + \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge {}_{123}\omega,$$

where the coefficients ${}_{\bullet}\omega$ are *horizontal forms*, that is, they verify $i_X({}_{\bullet}\omega) = 0$ for each $X \in \mathfrak{X}_{\Phi}(M)$. This is the *canonical decomposition* of ω. For example, $d\beta = {}_0(d\beta) + \chi_1 \wedge L_{X_1}\beta + \chi_2 \wedge L_{X_2}\beta + \chi_3 \wedge L_{X_3}\beta$, for any horizontal form β. Since $L_{X_i}\chi_j = \chi_{[u_i,u_j]}$, with $1 \le i, j \le 3$, then

$$L_{X_1}\chi_1 = L_{X_2}\chi_2 = L_{X_3}\chi_3 = 0, \qquad L_{X_1}\chi_2 = -L_{X_2}\chi_1 = \chi_3$$
$$L_{X_1}\chi_3 = -L_{X_3}\chi_1 = -\chi_2 \qquad L_{X_2}\chi_3 = -L_{X_3}\chi_2 = \chi_1. \tag{4}$$

and we have the canonical decompositions

$$\begin{cases} d\chi_1 &=& e_1 - \chi_2 \wedge \chi_3 \\ d\chi_2 &=& e_2 + \chi_1 \wedge \chi_3 \\ d\chi_3 &=& e_3 - \chi_1 \wedge \chi_2. \end{cases} \tag{5}$$

Consider $U \subset M\backslash\Sigma$ an equivariant open subset. If $\omega \in \Omega^*(M\backslash\Sigma, U)$ then the coefficients of its canonical decomposition are horizontal forms of $\Omega^*(M\backslash\Sigma, U)$. The following Lemma is the key for the construction of the Gysin sequence. Given an action of $\mathbb{Z}_2$ on a vector space E generated by the morphism $h\colon E \to E$, we shall write

$$E^{-\mathbb{Z}_2} = \{e \in E \mid h(e) = -e\}, \tag{6}$$

the subspace of *antisymmetric elements*. Notice that $j \in \mathbb{S}^3$ acts naturally on $M^{\mathbb{S}^1}$.

[5]Notice that this is not the five lemma.

[6]All the isotropy subgroups are finite groups.

Lemma 2.2

$$H^*\left(\frac{\underline{\Omega}^{\cdot}_V(M)}{\Omega^{\cdot}_V\left(M/\mathbb{S}^3\right)}\right) = H^{*-3}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

Proof. We consider the integration operator:

$$\fint : \frac{\underline{\Omega}^*_V(M)}{\Omega^*_V\left(M/\mathbb{S}^3\right)} \longrightarrow \Omega^{*-3}_V\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right),$$

given by:

$$\fint(<\omega>) = (-1)^{\deg\omega}\, i_{X_3} i_{X_2} i_{X_1}\omega.$$

It is a well defined differential operator since

- the tubular neighborhoods of the Thom-Mather's structure $\mathfrak{T}$ are invariant,
- the operator $i_{X_3} i_{X_2} i_{X_1}$ vanishes on Σ, and
- $i_X i_{X_3} i_{X_2} i_{X_1}\omega = i_X d i_{X_3} i_{X_2} i_{X_1}\omega = 0$ for each $X \in \mathfrak{X}_\Phi(M)$[7].

Every form $\gamma \in \Omega^{*-3}_V\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)$ vanishes in a neighborhood of Σ. So, the product $\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \gamma$ belongs to $\underline{\Omega}^*_V(M)$ (cf. (4)). Since $i_{X_3} i_{X_2} i_{X_1}(\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \gamma) = \gamma$ then we have the short exact sequence

$$0 \longrightarrow \operatorname{Ker}^* \fint \hookrightarrow \frac{\underline{\Omega}^*_V(M)}{\Omega^*_V\left(M/\mathbb{S}^3\right)} \xrightarrow{\fint} \Omega^{*-3}_V\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right) \longrightarrow 0 \tag{7}$$

By Lemma 1.7, it suffices to prove the following:

(a) $H^*\left(\operatorname{Ker}^* \fint\right) = \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$.

(b) The associated connecting homomorphism δ vanishes.

(a)

For the sake of simplicity we put $\mathcal{A}^*(M) = \operatorname{Ker}^* \fint$. In fact we have $\mathcal{A}^*(M) = \dfrac{\left\{\omega \in \underline{\Omega}^*_V(M) \mid i_{X_3} i_{X_2} i_{X_1}\omega = 0\right\}}{\Omega^*_V\left(M/\mathbb{S}^3\right)}$.
Analogously, we define $\mathcal{A}^*(M,\Sigma)$, $\mathcal{A}^*(\Sigma)$ and $\mathcal{A}^*(\Sigma, F)$. To get (a), it suffices to prove the following facts:

(a1) $H^*\left(\mathcal{A}^*(M)\right) = H^*\left(\mathcal{A}^*(\Sigma)\right)$.

(a2) $H^*(\mathcal{A}^{\cdot}(\Sigma)) = H^*(\mathcal{A}^{\cdot}(\Sigma, F))$.

(a3) $H^*(\mathcal{A}^{\cdot}(\Sigma, F)) = \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$.

(a1)

[7] $L_A i_B = i_B L_A + i_{[A,B]}, \quad \forall A, B \in \mathfrak{X}(M)$.

Consider the inclusion $L\colon \mathcal{A}^*(M,\Sigma) \longrightarrow \mathcal{A}^*(M)$ and the restriction $R\colon \mathcal{A}^*(M) \to \mathcal{A}^*(\Sigma)$, which are differential morphisms. This gives the short sequence

$$0 \longrightarrow \mathcal{A}^*(M,\Sigma) \xrightarrow{L} \mathcal{A}^*(M) \xrightarrow{R} \mathcal{A}^*(\Sigma) \longrightarrow 0.$$

Notice that $R{\circ}L = 0$. This short sequence is exact since:

$\bullet$*The operator R is an onto map.* Consider $\gamma \in \underline{\Omega}^*_V(\Sigma)$. We know that $\sigma(\gamma) \in \Omega^*_V(M)$ (cf. Lemma 1.4). The result comes from:

$\rightsquigarrow \sigma(\gamma) \in \underline{\Omega}^*_V(M)$. Since τ_0, τ_1 are equivariant and $f{\circ}\nu_0$, $f{\circ}\nu_1$ are invariant.

$\rightsquigarrow i_{X_3} i_{X_2} i_{X_1} \sigma(\gamma) = 0$. Since τ_0, τ_1 are equivariant and rank $\{X_1(x), X_2(x), X_3(X)\} \le 2$ for any $x \in \Sigma$.

$\rightsquigarrow R\,(< \sigma(\gamma) >) =< (\sigma(\gamma))_1 >=< \gamma >$.

$\bullet$ Ker $R \subset$ Im L. Consider $\omega \in \underline{\Omega}^*_V(M)$ with $i_{X_3} i_{X_2} i_{X_1} \omega = 0$ and $i_{X_j}\omega_1 = 0$ for $j \in \{1,2,3\}$. Since τ_0 and τ_1 are equivariant and $X_j = 0$ on F then $i_{X_j}\sigma(\omega_1) = 0$ for $j \in \{1,2,3\}$. This gives $< \sigma(\omega_1) >= 0$. Finally, we have $< \omega >=< \omega - \sigma(\omega_1) >= L\,(< \omega - \sigma(\omega_1) >)$ since $(\omega - \sigma(\omega_1))_1 = \omega_1 - (\sigma(\omega_1))_1 = \omega_1 - \omega_1 = 0$.

Now, we will get (a1) by proving that $H^*(\mathcal{A}^{\cdot}(M,\Sigma)) = 0$. By definition of Verona's forms we have $\mathcal{A}^*(M,\Sigma) = \mathcal{A}^*(M,D) \overset{excision}{=\!=\!=} \mathcal{A}^*(M\backslash\Sigma, D\backslash\Sigma)$, where $D = D_0 \cup D_1$. A straightforward calculation gives:

$$H^*\Big(\mathcal{A}^{\cdot}(M\backslash\Sigma, D\backslash\Sigma)\Big) = \frac{\Big\{\omega \in \underline{\Omega}^*(M\backslash\Sigma, D\backslash\Sigma) \,\Big|\, i_{X_3} i_{X_2} i_{X_1}\omega = 0 \text{ and } i_{X_j} d\omega = 0 \text{ for } j \in \{1,2,3\}\Big\}}{\Omega^*\Big((M\backslash\Sigma)/\mathbb{S}^3, (D\backslash\Sigma)/\mathbb{S}^3\Big) + \Big\{d\beta \,\Big|\, \beta \in \underline{\Omega}^{*-1}(M\backslash\Sigma, D\backslash\Sigma) \text{ and } i_{X_3} i_{X_2} i_{X_1}\beta = 0\Big\}}$$

Let ω be a differential form of $\underline{\Omega}^*(M\backslash\Sigma, D\backslash\Sigma)$ verifying $i_{X_3} i_{X_2} i_{X_1}\omega = 0$ and $i_{X_j} d\omega = 0$ for $j \in \{1,2,3\}$. Then

$$\omega = \begin{array}{c} d\,\underbrace{(\chi_1 \wedge i_{X_3} i_{X_2}\omega - \chi_2 \wedge i_{X_3} i_{X_1}\omega + \chi_3 \wedge i_{X_2} i_{X_1}\omega)}_{\beta} \\ + \\ \underbrace{-e_1 \wedge i_{X_3} i_{X_2}\omega + e_2 \wedge i_{X_3} i_{X_1}\omega - e_3 \wedge i_{X_2} i_{X_1}\omega + {}_0\omega}_{\alpha} \end{array}$$

(cf. (5)) with $\beta \in \underline{\Omega}^{*-1}(M\backslash\Sigma, D)$, verifying $i_{X_3} i_{X_2} i_{X_1}\beta = 0$, and $\alpha \in \Omega^*\Big((M\backslash\Sigma)/\mathbb{S}^3, D/\mathbb{S}^3\Big)$. This implies $H^*(\mathcal{A}^{\cdot}(M\backslash\Sigma, D\backslash\Sigma)) = 0$ and then $H^*(\mathcal{A}^{\cdot}(M,\Sigma)) = 0$.

(a2)

Consider the inclusion $L\colon \mathcal{A}^*(\Sigma, F) \hookrightarrow \mathcal{A}^*(\Sigma)$ which is a differential morphism. It suffices to prove that L is an onto map.

Let us consider a smooth function $f\colon]0,\infty[\to [0,1]$ verifying $f \equiv 0$ on $[3,\infty[$ and $f \equiv 1$ on $]0,2]$. Notice that the composition $f{\circ}\nu_0\colon M \to [0,1]$ is a smooth invariant map. So, for each $\gamma \in \Omega^*(F)$ we have $\sigma(\gamma) = (f{\circ}\nu_0)\tau_0^*\gamma \in \Omega^*(M)$. This differential form verifies

$\rightsquigarrow \sigma(\gamma) \in \Omega^*_V(\Sigma)$. Since $(f{\circ}\nu_0) \equiv 1$ on Δ_0 then $\sigma(\gamma) = \tau_0^*\gamma$ on $\Delta_0\backslash F$. This gives $(\sigma_0(\gamma))_0 = \gamma$.

$\rightsquigarrow \sigma(\gamma) \in \underline{\Omega}^*_V(\Sigma)$. Since τ_0 is an equivariant map and $f{\circ}\nu_0$ is an invariant map.

$\rightsquigarrow i_{X_j}\sigma(\gamma) = 0$ for $j \in \{1,2,3\}$ since τ_0 is an equivariant map and $X_j = 0$ on F.

Then $< \sigma(\gamma) >= 0$ on $\mathcal{A}^*(\Sigma)$.

Let $< \omega >$ be a class of $\mathcal{A}^*(\Sigma)$. We can write: $< \omega >=< \omega - \sigma((\omega)_0) >= L\,(< \omega - \sigma((\omega)_0) >)$ since $(\omega - \sigma((\omega)_0))_0 = \omega_0 - (\sigma(\omega_0))_0 = \omega_0 - \omega_0 = 0$. This proves that L is an onto map.

(a3)

By definition of Verona's differential forms we have

$$\mathcal{A}^*(\Sigma, F) = \mathcal{A}^*(\Sigma, \Delta_0) \overset{excision}{=\!=} \mathcal{A}^*(\Sigma\backslash F, \Delta_0\backslash F) = \frac{\underline{\Omega}^*(\Sigma\backslash F, \Delta_0\backslash F)}{\Omega^*\left((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3\right)}.$$

The isotropy subgroup of a point of $\Sigma\backslash F$ is conjugated to $\mathbb{S}^1$ or $N(\mathbb{S}^1)$ (cf. [3, Th. 8.5, pag. 153]). We consider the manifold $\Gamma = (\Sigma\backslash F)^{\mathbb{S}^1}$. A straightforward calculation gives that $\Sigma\backslash F$ is G-equivariant diffeomorphic to

$$\mathbb{S}^3 \times_{N(\mathbb{S}^1)} \Gamma = \left(\mathbb{S}^3/\mathbb{S}^1\right) \times_{N(\mathbb{S}^1)/\mathbb{S}^1} \Gamma = \mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma.$$

Notice that $\Gamma/\mathbb{Z}_2 = (\Sigma\backslash F)/\mathbb{S}^3$. Put Γ_0 the open subset $\Gamma \cap \Delta_0$ of Γ. Analogously we have $\Delta_0\backslash F = \mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma_0$ and $\Gamma_0/\mathbb{Z}_2 = (\Delta_0\backslash F)/\mathbb{S}^3$.

The $\mathbb{Z}_2$-action on $\mathbb{S}^2$ is generated by $(x_0, x_1, x_2) \mapsto (-x_0, -x_1, -x_2)$[8]. Then, the $\mathbb{Z}_2$-action on $H^0\left(\mathbb{S}^2\right)$ (resp. $H^2\left(\mathbb{S}^2\right)$) is the identity Id (resp. $-\mathrm{Id}$). The $\mathbb{Z}_2$-action on Γ is induced by $\Phi(j, -)$. The Künneth formula gives

$$\begin{aligned}
H^*\left(\underline{\Omega}^*(\Sigma\backslash F, \Delta_0\backslash F)\right) &= H^*\left(\underline{\Omega}^{\cdot}\left(\mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma, \mathbb{S}^2 \times_{\mathbb{Z}_2} \Gamma_0\right)\right) = H^*\left(\underline{\Omega}^{\cdot}\left(\mathbb{S}^2 \times \Gamma, \mathbb{S}^2 \times \Gamma_0\right)^{\mathbb{Z}_2}\right) \\
&= H^*\left(\underline{\Omega}^{\cdot}\left(\mathbb{S}^2 \times \Gamma, \mathbb{S}^2 \times \Gamma_0\right)\right)^{\mathbb{Z}_2} = \left(H^*\left(\mathbb{S}^2\right) \otimes H^*(\Gamma, \Gamma_0)\right)^{\mathbb{Z}_2} \\
&= \left(H^0\left(\mathbb{S}^2\right) \otimes H^*(\Gamma, \Gamma_0)\right)^{\mathbb{Z}_2} \oplus \left(H^2\left(\mathbb{S}^2\right) \otimes H^{*-2}(\Gamma, \Gamma_0)\right)^{\mathbb{Z}_2} = \left(H^*(\Gamma, \Gamma_0)\right)^{\mathbb{Z}_2} \oplus \left(H^{*-2}(\Gamma, \Gamma_0)\right)^{-\mathbb{Z}_2} \\
&= H^*(\Gamma/\mathbb{Z}_2, \Gamma_0/\mathbb{Z}_2) \oplus \left(H^{*-2}(\Gamma, \Gamma_0)\right)^{-\mathbb{Z}_2} = H^*\left((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3\right) \oplus \left(H^{*-2}(\Gamma, \Gamma_0)\right)^{-\mathbb{Z}_2},
\end{aligned}$$

and then

$$\begin{aligned}
H^*\left(\mathcal{A}^{\cdot}(\Sigma\backslash F, \Delta_0\backslash F)\right) &= H^*\left(\frac{\underline{\Omega}^{\cdot}(\Sigma\backslash F, \Delta_0\backslash F)}{\Omega^{\cdot}\left((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3\right)}\right) = \left(H^{*-2}(\Gamma, \Gamma_0)\right)^{-\mathbb{Z}_2} = \left(H^{*-2}\left((\Sigma\backslash F)^{\mathbb{S}^1}, (\Delta_0\backslash F)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \\
&\overset{excision}{=\!=} \left(H^{*-2}\left(\Sigma^{\mathbb{S}^1}, \Delta_0^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \overset{retraction}{=\!=} \left(H^{*-2}\left(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}.
\end{aligned}$$

Consider the long exact sequence associated to the $\mathbb{Z}_2$-invariant pair $\left(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1}\right)$:

$$\cdots \to \left(H^{i-1}\left(F^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \left(H^i\left(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \left(H^i\left(\Sigma^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \left(H^i\left(F^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \cdots.$$

Since the action of $\mathbb{Z}_2$ on $F^{\mathbb{S}^1} = F$ is trivial, then $\left(H^i\left(F^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = 0$. On the other hand, we have $\Sigma^{\mathbb{S}^1} = M^{\mathbb{S}^1}$. This gives $\left(H^{*-2}\left(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$.

(b)

Notice that the connecting morphism δ is defined by $\delta([\zeta]) = \pm[< d(\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3) \wedge \zeta >]$. We have $\delta \equiv 0$ since $\zeta_1 = 0$ (cf (a1)). ♣

[8] This map is induced by $j \colon \mathbb{S}^3 \to \mathbb{S}^3$ defined by $j(u) = u \cdot j$ (see [2, Example 17.23]).

Theorem 2.3 *Given any smooth action $\Phi\colon \mathbb{S}^3 \times M \longrightarrow M$ we have the Gysin sequence*

$$\cdots \longrightarrow H^i(M) \longrightarrow H^{i-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{i-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \longrightarrow H^{i+1}\left(M/\mathbb{S}^3\right) \longrightarrow H^{i+1}(M) \longrightarrow \cdots$$

where Σ is the subset of points of M whose isotropy group is infinite, the $\mathbb{Z}_2$-action is induced by $j \in \mathbb{S}^3$ and $(-)^{-\mathbb{Z}_2}$ denotes the subspace of antisymmetric elements.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega^*_{V}\left(M/\mathbb{S}^3\right) \longrightarrow \underline{\Omega}^*_{V}(M) \longrightarrow \frac{\underline{\Omega}^*_{V}(M)}{\Omega^*_{V}\left(M/\mathbb{S}^3\right)} \longrightarrow 0, \tag{8}$$

take its associated long exact sequence and then, apply Lemma 1.7, (3) and Lemma 2.2. ♣

2.4. Example.
Consider the connected sum $M = \mathbb{CP}^2 \# \mathbb{CP}^2 \cong \left(\mathbb{S}^3 \times [0,1]\right)/\sim$, with

$$((z_1, z_2), i) \sim ((z \cdot z_1, z \cdot z_2), i), \quad i = 0, 1,$$

for all $z \in \mathbb{S}^1$ and $(z_1, z_2) \in \mathbb{S}^3$ in complex coordinates. The product of $\mathbb{S}^3$ induces on M the action:

$$g \cdot [h, t] = [g \cdot h, t], \quad \forall g, h \in \mathbb{S}^3, \forall t \in [0,1].$$

For this action, we have:

$$\Sigma = \left(\mathbb{S}^3 \times \{0,1\}\right)/\sim \;\cong \mathbb{S}^2 \times \{0,1\}, \qquad F = \emptyset,$$

$$M/\mathbb{S}^3 \cong [0,1], \quad \Sigma/\mathbb{S}^3 \cong \{0,1\}, \quad M^{\mathbb{S}^1} \cong \{N, S\} \times \{0,1\},$$

where N and S stand for the North and South poles of $\mathbb{S}^2$. The $\mathbb{Z}_2$- action on $M^{\mathbb{S}^1}$ is determined by $j \in \mathbb{S}^3$, which induces the antipodal map on $\mathbb{S}^2$, and so, interchanges its poles. Thus, the exotic term that appears in the central part of the Gysin Sequence is not trivial:

$$H^2(M) \xrightarrow{\cong} \left(H^0\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \left(H^0(\{N,S\} \times \{0,1\})\right)^{-\mathbb{Z}_2} \cong \mathbb{R} \oplus \mathbb{R}.$$

2.5. Morphisms. We describe the morphisms of the Gysin sequence .

$$①\colon H^*\left(M/\mathbb{S}^3\right) \longrightarrow H^*(M)$$

It is the pull-back π^* of the canonical projection $\pi\colon M \to M/\mathbb{S}^3$ (cf. Lemma 1.7).

$$②\colon H^*(M) \longrightarrow H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

We have already seen that the first component of this morphism is induced by $⨍_{\mathbb{S}^3}[\omega] = [i_{X_3} i_{X_2} i_{X_1} \omega]$. For the second component we keep track of the isomorphisms given by Lemma 2.2 and we get that it is defined by: $[\omega] \mapsto$ class $\left(⨍_{\mathbb{S}^2}(\omega_1 - \sigma(\iota^*\omega_1))\right)$.

$$③\colon H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \longrightarrow H^{*+1}\left(M/\mathbb{S}^3\right)$$

A straightforward calculation using sequences (7) and (8) gives that the connecting morphism ③ of the Gysin sequence sends:

- $[\zeta] \in H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right)$ to -$\left[\left(e_1^2 + e_2^2 + e_3^2\right) \wedge \zeta\right]$, and
- $[\xi] \in \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \left(H^{*-2}\left(\Sigma^{\mathbb{S}^1}, F^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$ to $[d\sigma \wedge \epsilon \wedge \tau_1^* \xi]$ where ϵ is an Euler form of the restriction $\Phi_1 \colon \mathbb{S}^1 \times \left(\tau_1^{-1}\left(\Sigma^{\mathbb{S}^1}\right) \backslash \Sigma^{\mathbb{S}^1}\right) \to \left(\tau_1^{-1}\left(\Sigma^{\mathbb{S}^1}\right) \backslash \Sigma^{\mathbb{S}^1}\right)$ of Φ.

Since $e_1^2 + e_2^2 + e_3^2$ is not a Verona's form, then it does not define a class of $H^4\left(M/\mathbb{S}^3\right)$. Nevertheless, it does generate a class in the intersection cohomology group $\boldsymbol{H}^4_{\bar{4}}\left(M/\mathbb{S}^3\right)$ (as in the semi-free case of [6]).

2.6. Remarks.

(a) We have $\left(H^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = H^*\left(M^{\mathbb{S}^1}\right) / H^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right)$. Let us see that. The correspondence $\omega \mapsto \left(\dfrac{\omega + j^*\omega}{2}, \dfrac{\omega - j^*\omega}{2}\right)$ establishes the isomorphism $\Omega^*\left(M^{\mathbb{S}^1}\right) = \left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2} \oplus \left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \Omega^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right) \oplus \left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$ and hence, $H^*\left(M^{\mathbb{S}^1}\right) = H^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right) \oplus \left(H^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$. This gives the claim.

(b) Let us suppose that the action is semi-free, almost free or free. Then, j acts trivially on $M^{\mathbb{S}^1} = F$, and hence, we have a long exact sequence

$$\cdots \to H^i(M) \to H^{i-3}\left(M/\mathbb{S}^3, F\right) \to H^{i+1}\left(M/\mathbb{S}^3\right) \to H^{i+1}(M) \to \cdots .$$

(c) Let us suppose that there is not a point of M whose isotropy subgroup is conjugated to $\mathbb{S}^1$. Then, we have a long exact sequence

$$\cdots \to H^i(M) \to H^{i-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \to H^{i+1}\left(M/\mathbb{S}^3\right) \to H^{i+1}(M) \to \cdots .$$

since j acts trivially on $M^{\mathbb{S}^1} = \left\{x \in M \mid \mathbb{S}^3_x = \mathbb{S}^3 \text{ or } N(\mathbb{S}^1)\right\}$.

2.7. Actions over $\mathbb{S}^1$.

Using the Gysin sequence we have constructed, we now give a list of all the different cohomologies of a $\mathbb{S}^3$-manifold M having the circle as orbit space[9]. By geometrical reasons, the orbit space is composed by just one stratum, the whole circle. Following the nature of the orbits, we distinguish four cases.

(a) All orbits are of dimension 3. We have $P_M = 1 + t + t^3 + t^4$. This is the case of the manifold $\mathbb{S}^3 \times \mathbb{S}^1$, where $\mathbb{S}^3$ acts by multiplication on the left factor.

(b) All orbits are isomorphic to $\mathbb{S}^2$. We distinguish two cases following wether the covering $M^{\mathbb{S}^1} \to M^{\mathbb{S}^1}/\mathbb{Z}_2 = M/M^{\mathbb{S}^1}$ is trivial or not. In the first case we have $P_M = 1 + t + t^2 + t^3$. This is the case of the manifold $\mathbb{S}^2 \times \mathbb{S}^1$, where $\mathbb{S}^3$ acts by multiplication on the left factor. In the second case we have $P_M = 1 + t$, as is the case of the manifold $\mathbb{S}^2 \times_{\mathbb{Z}^2} \mathbb{S}^1$ where $\mathbb{S}^3$ acts by multiplication on the left factor.

(c) All orbits are isomorphic to $\mathbb{RP}^2$. In this case, we have $P_M = 1 + t$. This is the case of the manifold $\mathbb{RP}^2 \times \mathbb{S}^1$ where $\mathbb{S}^3$ acts by multiplication on the left factor.

(d) All orbits are points. We have $P_M = 1 + t$. This corresponds to the manifold $\mathbb{S}^1$ where $\mathbb{S}^3$ acts ineffectively.

[9] In fact, we give the Poincaré polynomial P_M of M.

References

[1] J.A. Álvarez López: *Duality in the spectral sequence of Riemannian foliations.* - Amer. J. of Math. **111** (1989), 905–926.

[2] R. Bott and L. W. Tu: Differential forms in Algebraic Topology. -Graduate Texts in Math., Springer Verlag, New York, Heidelberg, Berlin, 1986.

[3] G. Bredon: Introduction to compact transformation groups. - Pure and Appl. Math., Academic Press, New York and London, 1972.

[4] W. Greub, S. Halperin and R.Vanstone: Connections, curvature and cohomology - Pure and Appl. Math., vol. II, Academic Press, New York and London, 1972.

[5] J.I. Royo Prieto, Estudio Cohomológico de flujos riemannianos - Ph.D. Thesis, Universidad del País Vasco/Euskal Herriko Unibertsitatea, 2003.

[6] M. Saralegi: *A Gysin sequence for semi-free actions of* $\mathbb{S}^3$. - Proceedings of the A.M.S. **38** (1993), 1335–1345.

[7] A. Verona: *Le théorème de de Rham pour les prestratifications abstraites.* - Comp. Rend. Acad. Sci. Paris **273** (1971), 886–889.

[8] A. Verona: *Stratified Mappings-Structures and triangulability.* Lecture Notes in Math. **1102** Springer Verlag, 1984.

[9] C.T. Yang : *The triangulability of the orbit space of a differentiable transformation group.* - Bull. Am. Math. Soc. **69** (1963), 405–408.